

A Comparative Study of Exact Solutions and Numerical Approximation Techniques for Second Order Linear Ordinary Differential Equations

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ABSTRACT

This study examines numerical approximation approaches and precise solutions to second-order linear ordinary differential equations (ODEs) in terms of convergence stability, accuracy, efficiency, and convergence rate. The research compares numerical solutions derived from the Modified Euler Method, the Fourth Order Runge-Kutta Method, and the characteristic equation, as well as those derived from the variation of parameters method and the undetermined coefficients method. The computational complexity, absolute and relative error, how quickly the solution converges, and the impact of step size are all looked at. The numerical results are then checked against the exact analytical solutions to see how accurate they are. The comparison is based on theory, and it shows that the Euler Method, Modified Euler Method, and Fourth-Order Runge-Kutta Method have global accuracy of first order, second order, and fourth order respectively, as long as certain smoothness and stability conditions are met. The study highlights the trade-off between computation effort and accuracy, and provides a systematic approach to identifying the different solution approaches, why they can be used, their advantages and disadvantages. But other computational validations with different differential equations and step sizes need to be performed to validate their actual numerical performances.

Keywords: Ordinary Differential Equations, Exact Analytical Solutions, Numerical Approximation, Euler Method, Modified Euler Method, Runge-Kutta Method, Error Analysis, Convergence.

1. INTRODUCTION

A second-order linear ordinary differential equation (ODE) is a basic mathematical equation that describes many physical, engineering, and scientific phenomena. They have a wide range of applications in modelling mechanical vibrations, electrical circuits, heat transfer, fluid dynamics, population systems etc. They are useful in representing relationships between a function and its first

and second derivatives, which enables their use in analysing dynamic systems and understanding complex mathematical processes.

Analytical solutions to differential equations give exact mathematical expressions and are used to gain insight into the behaviour, properties and theoretical characteristics of mathematical models. There are several analytical methods that can be used to get exact solutions: characteristic equation method, undetermined coefficients, and variation of parameters. For many equations, however, exact solutions may not be found, especially if the equations have variable coefficients, complicated forcing functions, or complicated boundary and initial conditions.

If exact analytical solutions of differential equations are difficult or impossible to obtain, alternative approaches to their solution can be provided by numerical approximation techniques. Evaluating these methods by means of these features is important, because depending on these features, they vary in their accuracy, convergence, computational complexity and stability, which makes it important to evaluate these methods comparatively, in order to apply methods to different mathematical problems.

In the present study exact analytical methods are investigated and the performance of some numerical approximation methods is compared. The study involves approximate solution and exact analytical solution to compare the numerical accuracy, computational efficiency, numerical convergence behaviour and stability. The results will help create a better understanding of the advantages, disadvantages, and how to use different methods for solving second order linear ordinary differential equations.

1.1. Research Objectives

The research objectives of the study are:

- To explore precise analytical techniques used to solve second order linear ODE's.
- To compare numerical approximation techniques with exact solutions to see how accurate, efficient and convergent they are.

2. REVIEW OF LITERATURE

Arefin et al. (2020) compared the various numerical methods to solve ordinary differential equations. They investigated the accuracy, efficiency and behaviour of the numerical methods. The researchers emphasized that the use of proper numerical methods is a crucial step depending on the property of the differential equations. The result was a comparative analysis of numerical methods and the development of the use of computational methods to solve differential equations.

Denis (2020) provided a synopsis of analytical and numerical approaches to solving ODEs. The study covered the various methods of finding both approximate and exact solutions and considered the principles and applications of approximations. It emphasized how analytical techniques can help to find exact solutions and how numerical techniques can be used for problems where exact solutions were not readily available. The study includes a theoretical basis on advantages and drawbacks of employing various solution techniques.

Fairweather and Meade (2020) studied the use of approximation techniques based on splines and their mathematical background and computational features. The study highlighted the relevance of approximation methods for approximating solutions of differential equations numerically and showed the relevance of spline collocation in the computational mathematics. Their review helped to enhance knowledge of other numerical methods that would help to be more accurate and efficient in solving differential equations.

Ma et al. (2021) compared the automatic differentiation and continuous sensitivity analysis approaches to computing derivatives of solutions of differential equations. They investigated the methods of computation of sensitivities and they tested their efficacy for the solution of problems arising from differential equations. The researchers emphasized the importance of proper calculation of the derivatives and efficiency of the computation, when modeling mathematically. Their work led to the understanding of the use of the more sophisticated computational methods to analyze solutions of differential equations.

Zwillinger and Dobrushkin (2021) published a thorough reference book of differential equations in their Handbook of Differential Equations. Different types of differential equations, analytical solution techniques, mathematical formulation and different methods to solve problems of complex differential equations were studied. It gave the definitions and mathematical tools that were helpful for the investigation of exact solutions and numerical techniques. Their work was a handy reference book for the mathematical principles and methods to solve ordinary differential equations.

3. RESEARCH METHODOLOGY

This study employs a mathematical/computational method to compare exact analytical solutions with the numerical approximation methods for second order linear ODEs. It includes mathematical modelling, the use of analytical tools, such as the characteristic equation and variation of parameters,

and numerical tools, like Euler, Modified Euler and Runge-Kutta methods. The purpose of the study is to compare the performance, applicability and limitations of the various solution approaches.

3.1. Research Design

Mathematical and computational investigation into the best ways to approximate solutions to second-order linear ODEs numerically and analytically. Focusing on analytical and numerical approaches, this study compares the accuracy, efficiency, and convergence of the calculations as they pertain to solving the chosen differential equations.

3.2. Mathematical Formulation

This research is centred around the general form of second-order linear ordinary differential equations:

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

where $a(x)$, $b(x)$, and $c(x)$ are known coefficient functions, $f(x)$ is a given forcing function, and y is the unknown dependent variable. Selected initial-value problems with defined initial conditions will be used for mathematical analysis and computational comparison.

3.3. Exact Analytical Methods

The characteristic equation, method of indeterminate coefficients, and variation of parameters are well-established analytical techniques that will be applied wherever feasible to identify precise solutions to the chosen equations. These solutions will serve as benchmarks for evaluating how well the numerical approximation methods perform.

3.4. Numerical Approximation Techniques

Numerical solutions will be computed using selected numerical methods, including:

- **Euler Method:** To find approximate solutions by making step-by-step calculations.
- **Modified Euler Method:** To get a better approximation by adding slope corrections.
- **Fourth Order Runge-Kutta Method:** To obtain accurate numerical solutions by using higher order approximation.

To make meaningful comparisons, the numerical methods will be applied with consistent initial conditions and selected step sizes.

3.5. Comparative and Error Analysis

The performance of the numerical results will be compared with the exact results. The absolute error, relative error, convergence behaviour and computational efficiency will be taken into account in the study. The effects of different steps in accuracy and stability of numerical solutions will also be explored.

3.6. Computational Tools and Data Analysis

Appropriate programs for computation will be used to carry out mathematical computations and numerical simulations. Tabular and graphical representations of the exact and approximate solutions, distribution of errors and pattern of convergence will be used. The results will be analysed and interpreted to identify their accuracy, efficiency and limitations in terms of the numerical techniques employed.

3.7. Research Outcomes

The goal of the methodology is to establish a systematic comparison between analytic solutions and the methods of numerical approximation used for second order linear ODEs. The results will be compared with and contrasted the performance of the different techniques and will be used to determine which techniques are applicable to the solution of certain small examples of specific classes of differential equations.

4. RESULTS AND DISCUSSION

The section discusses the exact solution and numerical solutions to second order linear ODE with the Euler method, Modified Euler method and Fourth Order Runge-Kutta method. Compares these methods on the following parameters: accuracy, computational complexity, error analysis, convergence, efficiency and stability. Their accuracy is found to be different from each other, with varying computational expense, and the accuracy of using various numerical approximations is tested against the exact solution by theoretical results. The size of the errors and numerical values for them that are used to calculate their size are not yet known, the discussion is based on the theoretical properties and performance of the methods, and it serves as a basis for further numerical testing.

4.1. Exact and Numerical Solutions

The general second-order linear ordinary differential equation is expressed as:

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

To show the comparative approach, let's start with the following initial-value problem:

$$y'' + y = 0$$

with initial conditions:

$$y(0) = 0, y'(0) = 1$$

The exact analytical solution is:

$$y(x) = \sin x$$

The Euler, Modified Euler, and Fourth-Order Runge-Kutta methods are used to solve the problem analytically and to approximate it numerically. Each numerical solution's accuracy is evaluated using the precise solution.

4.2. Comparative Analysis of Numerical Methods

There are different computation procedures, accuracy of approximation and convergence rate for numerical methods. In Table 1 below, you can see the relative merits of three numerical approaches to solving an ODE. Below in Table 1 you can see a comparison of three numerical methods that are available for solving an ODE. All of the methods are assessed in terms of their accuracy, complexity, and convergence. The table below demonstrates the calculation of the procedure from the relatively simple Euler Method through to the more demanding Fourth Order Runge-Kutta Method. It provides a detailed account of theoretical aspects of each method in a structured way.

Table 1: Comparative Characteristics of Numerical Methods

Numerical Method	Order of Accuracy	Computational Complexity	General Convergence
Euler Method	$O(h)$	Low	First-order
Modified Euler Method	$O(h^2)$	Moderate	Second-order
Fourth-Order Runge-Kutta	$O(h^4)$	Higher per step	Fourth-order

Note: The orders shown refer to global error under suitable smoothness and stability conditions.

The comparison shows that the Euler Method has first-order accuracy and low computational complexity, making it a simple approach for numerical approximation. The Modified Euler Method has a better theoretical order of accuracy than Euler Method, and is more complex but still moderate. The Fourth Order Runge-Kutta Method is more accurate (order 4) and has a higher computational complexity per step (due to more calculations in each step). These findings show that there is a

compromise between the order of accuracy and computational effort. The higher order methods generally give better estimates for a fixed step size under good smoothness and stability conditions, although this may not always be the case for a specific differential equation and step size.

4.3. Error Analysis

Error analysis to measure the difference between exact analytical solutions and numerical approximations. The absolute error is given by:

$$E_a = |y_{\text{exact}} - y_{\text{numerical}}|$$

The relative error is expressed as:

$$E_r = \frac{|y_{\text{exact}} - y_{\text{numerical}}|}{|y_{\text{exact}}|}$$

These procedures are useful for determining the accuracy of numerical solutions at chosen points. If the exact solution is zero, then relative error is undefined and the absolute error may be used. The comparative error analysis of the Euler, Modified Euler, and Fourth Order Runge-Kutta methods for solving second-order linear ordinary differential equations is displayed in Table 2 below. Step size, absolute error, and relative error situations are used to compare the approaches. Relative error is the ratio of the difference to the exact answer, whereas absolute error is the difference between the numerical approximation and the actual solution. The numerical accuracy of each technique at the selected step size is shown in the table below.

Table 2: Comparative Error Analysis

Numerical Method	Step Size	Absolute Error	Relative Error
Euler Method	h	To be calculated	To be calculated
Modified Euler Method	h	To be calculated	To be calculated
Fourth-Order Runge-Kutta	h	To be calculated	To be calculated

Table 2 shows the mistakes made by the Fourth Order Runge-Kutta Method, the Modified Euler Method, and the Euler Method. Their actual mistakes can no longer be measured or compared in numbers. The Euler Method has first-order accuracy, the Modified Euler Method has second-order accuracy, and the Fourth-Order Runge-Kutta Method has fourth-order accuracy, and all of these are correct under specific conditions. Such differences mean that different error behaviours can be expected

if the same step size is used, but it is important to determine these differences, by calculation, before drawing conclusions about the comparative performance of the two different step sizes, when the step size is the same.

4.4. Convergence Analysis

Convergence analysis, the precise solution and the numerical solution are compared at different steps, and the convergence is noted as the step size is decreased. For the chosen techniques, the theoretical global error orders are first order with Euler, second order with Modified Euler, and fourth order with Runge-Kutta. Table 3 displays the convergence analysis framework for the Euler Method, Modified Euler Method, and Fourth-Order Runge-Kutta Method. It makes a comparison of the theoretical global error of each method and its expected behavior as the step size is decreased. The theory of the accuracy of the numerical methods is given and the framework explains what changes should be expected in the approximation error as the smallness of the steps is increased.

Table 3: Convergence Analysis Framework

Method	Theoretical Global Error	Expected Behaviour as h Decreases
Euler	$O(h)$	Linear error reduction
Modified Euler	$O(h^2)$	Quadratic error reduction
Fourth-Order Runge-Kutta	$O(h^4)$	Fourth-power error reduction

The convergence analysis indicates that the Euler Method has a theoretical global error of $O(h)$, the Modified Euler Method has a global error of $O(h^2)$, and the Fourth-Order Runge-Kutta Method has a global error of $O(h^4)$, where h represents the step size. The Euler Method is supposed to have linear error reduction, the Modified Euler Method has quadratic error reduction and the Fourth-Order Runge-Kutta Method is supposed to have fourth-power error reduction as the step size decreases. That is, if smoothness and stability conditions are appropriate and the step size is sufficiently large, then the global error should go down by about one-half, one-fourth, and one-sixteenth, respectively, when the step size is halved. These are only theoretical convergence properties; it is necessary to numerically test the convergence properties to prove the actual rate of error reduction for a specific problem.

4.5. Computational Efficiency and Stability

Table 4 gives a comparative study of Euler Method, Modified Euler Method and Fourth Order Runge-Kutta Method to calculate per step, order of accuracy, computational time per step, stability and suitability. The differences in the computational requirements and in theoretical accuracy of the three numerical methods are shown. It also describes in a general fashion the fields of their applications, from the simple to the ones needing a better approximation and higher accuracy solution.

Table 4: Computational Performance Comparison

Parameter	Euler	Modified Euler	Runge-Kutta
Calculation per step	Low	Moderate	High
Accuracy order	First	Second	Fourth
Computational time per step	Low	Moderate	Higher
Stability	Step-size dependent	Step-size dependent	Step-size dependent
Suitability	Simple problems	Improved approximations	Higher-accuracy solutions

The Euler Method needs few calculations per step, and has first order accuracy with a relatively small computational time per step, so it is good for simple problems. The Modified Euler Method is a second order method, for which the computational cost is moderate, and the approximation is more accurate at a higher cost. The Fourth-Order Runge-Kutta Method is more complex, with a greater number of calculations per step and has higher computational time per step, but it has 4th order accuracy, which is appropriate for problems for which higher accuracy solution is desired. All three are of step size dependent stability. Overall, the table shows the compromise that is often reached in numerical computational schemes between computational effort and theoretical accuracy, and the actual performance and suitability will depend on the nature of the differential equation and the choice of step size.

4.6. Discussion of Findings

The results of the study reveal that even though all the three methods were used to solve the same second-order linear ODE, there were theoretical differences with respect to accuracy, computational complexity and convergence between the Euler Method, Modified Euler Method, and Fourth Order

Runge-Kutta Method. The Euler Method has the advantage of first-order accuracy and low computational complexity, and can be used in simple problem solving. The Modified Euler Method has moderate accuracy of 2nd order and is more computationally efficient per step, whereas, the Fourth-Order Runge-Kutta Method is more computationally expensive per step and is 4th order accurate. The convergence analysis also suggests that for all three methods, the smaller the step size, the greater the degree of accuracy for the approximation, and that the higher order methods will normally have a faster rate of error decrease theoretically.

The comparative error analysis gives a framework for analysis of the absolute and relative errors of the numerical methods compared to the exact analytical solutions. The numerical performance of the methods, however, cannot be determined from the results, as the error values and the computational measurements have not yet been computed. These two methods are theoretically compared and tradeoffs in computational effort and accuracy are discussed depending on the complexity of the differential equation, the required accuracy, and the size of the steps used. The overall study offers a theoretical standpoint for comparison among various numerical methods of approximation with exact analytical solutions, with further numerical simulations needed to confirm the expected behaviour of the error, convergence, and efficiency.

5. CONCLUSION

In this study exact analytical solutions and numerical approximation technique to second order linear ordinary differential equations have been discussed and compared in terms of accuracy, computational efficiency, convergence and stability. Theoretical comparisons of the three methods showed that the Modified Euler Method was the most accurate at the second order with moderate computing cost, the 4th Order Runge-Kutta Method was the most accurate at the fourth order with higher computation cost, and the Euler Method was only accurate at the first order with low computation cost. The study demonstrates the balance between computation time and accuracy, depending on the type of the problem and the required accuracy. However, further computational testing of these methods using other equations and step sizes is required to establish their expected error behavior and performance.

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